



The Fictionalist Concept of Numbers: A Critique

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Abstract: The philosophy of mathematics is primarily concerned with the meaning of ordinary mathematical sentences and the question of whether abstract mathematical objects exist, with the objectives of strengthening the mathematical theory and aiding mathematics education. On the existence of mathematical objects, positions advanced have tended to crystallise in a number of oppositions. We find Platonists who believe in the existence of abstract mathematical entities, as opposed to fictionalists who deny the existence of such entities and attempt to strip mathematics of its abstract qualities. We also find realists who believe in the objective mind-independence of mathematical truth-values, who are opposed by various types of anti-realists or fictionalists. We investigate the ontological mathematical divide between mathematical fictionalists and mathematical realists using critical analysis as our method. This conflict includes the question of the existence of abstract mathematical objects and entities, as well as their epistemological justification, if they exist at all. We further upheld mathematical realism based on its ability to prove mathematical objectivity, apodictic, a priori knowledge and promote ease in mathematics education.

Keywords: fictionalist, concept, numbers.

Introduction

Philosophy has always maintained a fascination with mathematics. Broadly speaking, Shapiro maintained that "rationalism is the attempt to apply the methodology of mathematics to all discursive thinking, to science and philosophy in particular" (Shapiro, 2016, p. 1). The application of philosophical tools to the study of mathematics is aimed at addressing questions such as: What is mathematical and logical knowledge? And how are such kinds of knowledge obtained? Are there mathematical entities? What is the epistemic justification of mathematical entities? The importance of this questions is not only in the advancement of the disciplines of mathematics and philosophy but equally for the facilitation of mathematics education.

Most students skip mathematics classes not because of unwillingness to learn the subject but because of the apparent abstract nature of the discipline. Mathematics' concepts seem to parade no empirical reference and therefore exoteric. It would however have garnered attraction among students if it is a narrative about exoteric entities. But it is rather a demonstrative and an analytic presentation of abstract entities standing in relation to one another in some form of rigorous logical structure. The rigour of mathematics is scary to students.

So, it is one of the objectives of the philosophy of mathematics, being originally an offshoot of analytic philosophy, to bring mathematics to the doorsteps of students, through the clarification of mathematical concepts. One of the most practical approaches adopted by analytic philosophers to provide clarification for concepts is the method of demonstrating the relationship between concepts and their referents. That approach dominates the programmes of realism and fictionalism examined in this paper.

On the existence of mathematical entities, fictionalists refute the claim of mathematical realists

that mathematics is the science of numbers, sets, functions, etc., just as physical science is the study of ordinary physical objects, astronomical bodies, subatomic particles, and so on." That is, mathematics is about these things, and the way these things are is what makes mathematical statements true or false" (Maddy, 1990, p. 2). This view seems simple and straightforward. Why should anyone think otherwise? Fictionalists do. Thus, they question the epistemological possibility of mathematical objects. If actually they do exist, what is their epistemological groundings? These objects are not mind-dependent (their existence is not the result of our mental processes and linguistic practices), and they are abstract (that is, mathematical objects are causally inert and are not located in space-time). To the Platonists/Realists, mathematics is ultimately about mathematical objects and their relations, the structures they determine and the concepts that are introduced in order to describe them. But how can we form even reliable beliefs, let alone knowledge, of such an abstract realm? This is indeed the epistemological problem we face.

Using critical analysis as our method, we will investigate the opposing views on the existence of mathematical objects held by mathematical realists and fictionalists.

Platonic/realist Concept of Numbers

The epistemological justification for a reliable belief in the abstract realm of numbers was first offered by Platonism. According to the Platonic account of mathematics, "mathematical objects exist independently of us. These objects are not mind-dependent (their existence is not the result of our mental processes and linguistic practices), and they are abstract (that is, mathematical objects are causally inert and are not located in space-time)" (Bueno 2011, p. 385).

Plato remarkably stated the essence of numbers and arithmetic to the Guardians and ruling class in the *Republic*. To the Guardians, mathematics and arithmetic are a necessity. Since they are the philosophers' kings, they have to rise out of the sea of the changing physical world and lay hold of true being. Thus, arithmetic has a powerful and elevating effect on the soul by compelling it to reason about abstract numbers. Through arithmetic, we see the application of pure intelligence in the attainment of pure truth (Plato, 2013, p. 204). With this Platonic feat, we can say that "mathematics at least appears to produce necessary truths, and it has an exquisite certainty in its results" (Shapiro, 2016, p. 3). In our world today, those who accept this view are the realists who believe in the truth of mathematical statements and the possibilities of mathematical entities.

Indeed, the mathematical realists inspired by Plato "standardly endorse a face-value reading of those sentences used to express our mathematical and scientific theories and accept that such sentences so interpreted express truths. They therefore commit themselves to accepting the existence of mathematical objects" (Leng, 2018, IEP). For the Platonists, the knowledge and idea of numbers is primarily about mathematical objects and their relations, the structures they determine and the concepts that are introduced in order to describe them (Bueno, 2011).

According to Colyvan (2012), platonism is also known as mathematical realism. Mathematical realism advances the philosophical position that the statements; “‘there are infinitely many prime numbers’ is true by virtue of the existence of mathematical objects which are prime numbers”. To the Platonist, the statement ‘3 is prime’ offers an unambiguous picture of a mathematical object about the *number 3*. Balaguer (2018) remarked that the expression ‘3 is prime’ is similar to the statement ‘Mars is red’ which equally discloses a description of Mars. But whereas Mars is a physical object, the number 3 according to Platonism is an *abstract* object. To Platonists, abstract objects are wholly nonphysical, nonmental, nonspatial, nontemporal, and noncausal. Thus, on this view, the number 3 exists independently of us and our thinking, but it does not exist in space or time, it is not a physical or mental object, and it does not enter into causal relations with other objects. Consequently, mathematical realism or Platonism can be viewed from two perspectives (i) there exist abstract mathematical objects (i.e., non-spatiotemporal mathematical objects), (ii) our mathematical sentences and theories provide true descriptions of such objects.

The defining characteristic of mathematical realism is the principle of objectivity. Mathematical objectivity clearly states that mathematical propositions are objectively true. Their truthfulness is not defined by sociological conventions or by the opinions or beliefs of mathematicians (Colyvan, 2012). What qualifies the truthfulness of mathematical objectivity, since it cannot be determined by the cultural perspective of the mathematician, is the existence of mathematical objects. Thus, mathematical statements are objectively true, and are made true by the existence of mathematical objects (Colyvan, 2012, p. 42). The nature of mathematical objects is that they are abstract entities with no causal powers and no spatial or temporal locations. Indeed, Colyvan (2012) disclosed that "a consistent mathematical theory truly describes some part of the mathematical universe" (p.43). Consequently, through mathematical objects, mathematical realism describes the universe through an ontology that is fundamental in the field of philosophy of mathematics. Mathematical realism therefore is "the view that at least some mathematical objects do exist objectively. Also, in mathematical realism, mathematical objects are *prima facie* abstract, acausal, indestructible, eternal, and not part of space and time. Since mathematical objects share these properties with Platonic Forms, realism in ontology is sometimes called Platonism (Shapiro, 2005, p. 6).

The question that follows from the above consideration is, "if indeed there is a mathematical ontology," In response to this question, some philosophers like Frege, Russell, Gödel, Quine, Colyvan, Putnam, etc. endorsed mathematical ontology and at various points proffered a justification of mathematical objects and the existence of mathematical entities. These philosophers have a unified central thesis in their various claims, which point at justifying the existence of abstract mathematical objects (for instance, non-spatiotemporal mathematical objects), and our mathematical sentences and theories provide true descriptions of such objects (Balaguer, 2018).

However, if mathematical objects are in fact abstract and thus causally isolated from the mathematician, as the mathematical realists claim, then how is it possible to gain knowledge of them? How can we form reliable beliefs, let alone knowledge, of such an abstract realm? Consequently, this identified "epistemological deficiency of mathematical realists" inspired the emergence of *mathematical fictionalists* or *mathematical nominalism*. The following section will, in fact, be a critical examination of mathematical fictionalism.

The Fictionalist Concept of Numbers

In the words of Balaguer (2018), mathematical fictionalism (or *fictionalism*) is best thought of as a reaction to mathematical Platonism. The fictionalists are of the view that our mathematical sentences and theories do purport to be about abstract mathematical objects, as Platonism suggests, because there are no such things as abstract objects, and so our mathematical theories are not true (Balaguer, 2018). In Leng's view, "mathematical fictionalists accept a face-value reading of sentences uttered in the context of mathematical and ordinary empirical theorizing, but when those sentences are, on that reading, committed to the existence of mathematical objects, mathematical fictionalists do not accept those sentences to be true" (Leng, 2019). Thus, it is worth noting that fictionalists do not reject the semantic strength of the first thesis of the mathematical realist, but they only reject the ontological implication of the thesis. With this claim, fictionalism can be seen as "a version of *mathematical nominalism*, the view that there are no such things as mathematical objects" (Balaguer, 2018).

Accordingly, it is worth noting that mathematical fictionalists, while refusing to accept the truth of mathematical realism, do not reject mathematical propositions. They also do not want to restrict mathematicians from doing mathematics or empirical scientists from doing mathematically-infused empirical science. Rather, they deny the Platonist's ontological thesis that there exist abstract objects.

Balaguer (2018) presents the main argument for fictionalism as follows:

1. Mathematical sentences like '4 is even' should be read at face value; that is, they should be read as being of the form " Fa " and, hence, as making straightforward claims about the nature of certain objects; e.g., '4 is even' should be read as making a straightforward claim about the nature of the number 4. But
2. If sentences like "4 is even" are taken at face value, and if they are true, then there must be objects of the kinds that they are about; for example, if "4 is even" makes a straightforward claim about the nature of the number 4, and if this sentence is literally true, then there must be such a thing as the number 4. Therefore, from (1) and (2), it follows that
3. If sentences like '4 is even' are true, then there are such things as mathematical objects. But
4. If there are such things as mathematical objects, then they are abstract objects, i.e., non-spatiotemporal objects; for instance, if there is such a thing as the number 4, then it is an abstract object, not a physical or mental object. But
5. There are no such things as abstract objects. Therefore, from (4) and (5), by modus tollens, it follows that
6. There are no such things as mathematical objects. And so, from (3) and (6) by modus tollens, it follows that
7. Sentences like '4 is even' are not true (indeed, they're not true for the reason that fictionalists give, and so it follows that fictionalism is true).

Consequently, the argument presented above is a logical implication expressed in the form of modus tollens that denies the existence of mathematical entities. The easiest way to state this argument is thus: Given that P is true, it is logically permissible that Q exists. And it follows that Q does not exist, therefore P is not true. Thus, from this argument, nominalism/fictionalism's doctrine that there are no abstract entities is established (Field, 2016). Even though Hartry Field acknowledged that "the term 'abstract entity' may not be entirely clear, one thing that does seem clear is that such alleged entities as numbers, functions, and sets are abstract. That is, if they existed, they would be abstract" (Field, 2016, p. 1).

To Benacerraf (1983), "numbers could not be sets and numbers could not be objects at all, for there is no more reason to identify any individual number with any one particular object than with any other (not already known to be a number)" (p. 291). With this submission, Benacerraf is of the view that numbers are not objects at all, because in giving the properties that are necessary and sufficient for numbers, one merely characterised an abstract structure, and the distinction lies in the fact that the elements of the structure have no properties other than those relating them to other elements of the same structure.

The thrust of this section has been an attempt to explore the fictionalists' concept of numbers. Through this approach, we identified fictionalists as nominalists, anti-Platonists, and anti-realists, with the aim of constructing a system of mathematics and/or physics that is consistent with their scepticism about abstract entities. Thus, we can equally say that "fictionalism" is an attempt to reproduce what ordinary mathematics has accomplished without presupposing the existence of such things as numbers and sets.

Critique of the Fictionalist Concept of Numbers

The underlying question that forms the critique of the fictionanlists' concept of numbers is: the ontological status of mathematical objects, i.e., do numbers, sets, and so on exist? This question has been the bone of contention in the philosophy of mathematics. Mathematical realists are of the view that mathematical entities such as functions, numbers, and sets have minds and exist independently (Colyvan, 2001). An important nominalist response to these arguments is fictionalism. "A fictionalist about mathematics believes that mathematical statements are, by and large, false. According to the fictionalist, mathematical statements are true in the story of

mathematics, but this does not amount to simple truth. "Fictionalists take their lead from some standard semantics for literary fiction" (Colyvan, 2001).

To attain mathematical knowledge, a suitable mathematical ontology is necessary. The idea is that once the ontology of mathematics is properly worked out, mathematical epistemology will turn out to be much less troublesome than it might initially seem to be. The major proponent of this view is Frege. To explicate this view, Bueno (2011) observes that:

The abstract character of mathematical objects emerges from the kind of thing these objects are: objects that fall under certain concepts. Given that Fregean concepts are abstract, mind-independent things, numbers and other mathematical objects inherit the same abstract character from the concepts they fall under. The Fregean Platonist has no difficulty making sense of the objectivity of mathematics, given the mind-independence of the concepts involved in their characterization. Frege's own motivation to develop his proposal, which was initially developed to make sense of arithmetic, emerged from the need to provide a formulation of arithmetic in terms of logic (p. 359).

With the given claim, the Platonist Gottlob Frege launched a fierce assault on early formalism from many directions simultaneously, but the most penetrating arose from just this point. Such that, Maddy (1990) disclosed that "it isn't hard to see how various true statements of mathematics can help me determine how many bricks it will take to cover the back patio, but how can a meaningless string of symbols be any more relevant to the solution of real world problems than an arbitrary arrangement of chess pieces?" (p. 24)

Also, mathematical realists have always argued from the position of the indispensability of numbers. This is an argument proffered by Quine (1948, 1951), Putnam (1971), Resnik (1997), and Colyvan (2001). In their various works, one simple line of argument runs through them thusly: (i) mathematical sentences form an indispensable part of our empirical theories of the physical world, for example, our theories of physics, chemistry, and so on; (ii) we have good reasons for thinking that these empirical theories are true, i.e., that they give us accurate pictures of the world; therefore, (iii) we have good reasons to think that our mathematical sentences are true and, hence, that fictionalism is false.

Colyvan (2001) disclosed that "if apparent reference to some entity (or class of entities) is indispensable to our best scientific theories, then we ought to believe in the existence of it" (p. 7). Colyvan further argued that:

One of the most intriguing features of mathematics is its applicability to empirical science. Every branch of science draws upon large and often diverse portions of mathematics, from the use of Lie groups in quantum mechanics to the use of differential geometry in cosmology. It's not only the physical sciences that avail themselves of the services of mathematics, either. Biology, for instance, makes extensive use of differential equations and statistics. The role mathematics plays in these theories is also varied. Not only does mathematics help with empirical predictions, but it also allows for elegant and economical statements of many theories. Indeed, so important is the language of mathematics that it is hard to imagine how some theories could even be stated without it. Furthermore, looking at the world through mathematical eyes has, on more than one occasion, facilitated enormous breakthroughs in science (Colyvan, 2001, p. 6).

However, Field's firm response is built on the refutation of premise (i). He argues that mathematics is, in fact, *not* indispensable to empirical science. Field (2016) tried to establish this thesis by arguing that our empirical theories can be *nominalized*, such that they are reformulated in a way that avoids reference to, and existential quantification over, abstract objects. However, Field's claim has been criticised as being extremely controversial, and it is extremely difficult to establish because, presumably, one would have to carry out the nominalization for each of our empirical theories. Thus, the name "*hard-road fictionalism*" (Balaguer, 2018). Mark Balaguer also argued that "Field did not try to do this for all of our empirical theories. Rather, he tried to motivate his position by explaining how the nominalization would go for one empirical theory,

namely, Newtonian Gravitation Theory. Now, some argue that "even if Field's strategy works for this one theory, it may not work for other theories" (2018).

Another objection to fictionalism is centred on the idea that fictionalists cannot account for the *objectivity* of mathematics. Mathematical realists have attempted to show that it is an apparent fact about mathematical practise that there is some sort of objectivity at work in that practice. For instance, Balaguer (2018) pointed out that "there's an important difference in mathematics between sentences like " $2 + 2 = 4$ " and "3 is prime" on the one hand and " $2 + 2 = 5$ " and "3 is composite" on the other. There's obviously *some* sense in the first two sentences, but not the second two, which are "correct," "right," or "good." The most obvious thing to say here is that the first two sentences are *true*, whereas the latter two are *false*. However, fictionalists cannot say this; they're committed to saying that all four of these sentences are untrue. Thus, the question arises whether fictionalists have any adequate account of the objectivity of mathematics.

Aldo Antonelli (2016), in *Semantic Nominalism: How I Learned to Stop Worrying and Love Universals*, offered a novel view on abstraction principles in order to solve a traditional tension between different requirements: that the claims of science be taken at face value, even when involving putative reference to mathematical entities; and that referents of mathematical terms be identified and their possible relations to other objects specified. In his view, abstraction principles provide representatives for equivalence classes of second-order entities that are available provided the first- and second-order domains are in the equilibrium dictated by the abstraction principles and whose choice is otherwise unconstrained. Abstract entities are the referents of abstraction terms; such referents are, to some extent, indeterminate, but we can still quantify them, predicate identity or non-identity, etc. With this analysis, Antonelli (2016) revealed that our knowledge of abstract entities is limited but still substantial, such that we know whatever has to be true no matter how the representatives are chosen and what is true in all models of the corresponding abstraction principles.

On the other hand, the central claim of fictionalists (the denial of abstract mathematical objects) is defended by Paul Benacerraf (1983) in his work entitled "What Numbers Could Not Be." To Benacerraf:

Numbers are not objects at all, because by giving the properties (that are necessary and sufficient) of numbers, you merely characterise an abstract structure, and the distinction lies in the fact that the elements of the structure have no properties other than those relating them to other elements of the same structure...that a system of objects exhibits the structure of integers implies that the elements of that system have some properties not dependent on structure. It must be possible to individuate those objects independently of the role they play in that structure. But this is precisely what cannot be done with numbers. To be number 3 is no more or no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth. And to be number 4 is no more or no less than to be preceded by 3, 2, 1, and possibly 0, and to be followed by any... Any object can play the role of 3; that is, any object can be the third element in some progression. What is peculiar to 3 is that it defines that role—not by being the paradigm of any object that plays it, but by representing the relation that any third member of a progression bears to the rest of the progression (p. 291).

From Benacerraf's position on the characteristics of numbers, its bearing on arithmetic is that it is a science that elaborates the abstract structure that all progressions have in common.

Also, while fictionalists believe that sentences like " $3 + 2 = 5$ " are strictly speaking false, they nonetheless think they're "correct" in some sense of the term. What, then, is the fictionalist's *position* toward these sentences? Bas van Fraassen (1980), who endorses a similar view with respect to empirical science, says the standard fictionalist line here is that they *accept* sentences like " $3 + 2 = 5$ " without *accepting* them as true. Just how exactly acceptance should be defined is a matter of some controversy, but one obvious way to proceed here is to claim that

fictionalists *accept* a pure mathematical sentence S if and only if they believe that S is true in the story of mathematics.

However, Burgess and Rosen (1997) present arguments for the claim that there is no real difference between acceptance and belief because, roughly, (a) to believe something is just to be disposed to behave in certain ways, and (b) those who believe that $2 + 2 = 4$ and those who allegedly only accept that $2 + 2 = 4$ are presumably disposed to behave in exactly the same ways.

Conclusion

The philosophy of mathematics is a borderline discipline of fundamental importance to both mathematics and philosophy. Central to this discourse is the question of the existence of mathematical objects. In this work, we attempted to review the concerns of both parties, as well as the background and notion of numbers, with the goal of assessing the divided claims on the nature of mathematical objects. Indeed, realists believe in the objective mind-independence of mathematical statements' truth values, while anti-realists or constructivist views reject the possibility of mathematical objects.

Conclusively, we agree with the rationalists about trying to secure the *a priori* status of mathematical knowledge. Opposed by various brands of empiricists trying to ground mathematics on empirical evidence. We have (or have had, at least) foundationalists looking for the basic bricks of the mathematical edifice and anti-foundationalists approaching mathematics as fallible when not sociologically determined. The absolutely *a priori* character of the justification of mathematical beliefs entails that they have to be indefeasible.

Evidently, Ayer (1983) argued that "there is nothing mysterious about the apodeictic certainty of logic and mathematics. Our knowledge that no observation can ever refute the proposition $7 + 15 = 12$ depends simply on the fact that the symbolic expression $7 + 15$ is synonymous with oculist, and the same explanation holds good for every other *a priori* truth" (p. 326). With this claim, it is our position that mathematical entities and mathematical objectivity are indubitable.

References

1. Antonelli, A. (2016). Semantic Nominalism: How I Learned to Stop Worrying and Love Universals. In F. Boccuni and A. Sereni (Eds.), *Objectivity, Realism, and Proof FilMat Studies in the Philosophy of Mathematics*. Springer International Publishing Switzerland.
2. Balaguer, M.(2018). Fictionalism in the Philosophy of Mathematics. In Edward N. Zalta (Ed.) The Stanford Encyclopedia of Philosophy, <http://plato.stanford.edu/archives/fall2018/entries/fictionalism-mathematics>
3. Benacerraf, P. (1983). What Numbers could not Be. In P. Benacerraf and H. Putnam (Eds.), *Philosophy of Mathematics: Selected Readings (2nd ed.)*. London: Cambridge University Press.
4. Bueno, O. (2011). Logical and Mathematical Knowledge. In S. Bernecker and D. Pritchard (Eds.), *Routledge Companion to Epistemology* New York: Tailor and Francis e-Library.
5. Burgess, J. Rosen, G. (1997). *A Subject with No Object*, New York: Oxford University Press Field, H. (2016). *Science Without Number: A Defence of Nominalism* (2nd ed.). Oxford: Oxford University Press.
6. Leng, M. Fictionalism in the Philosophy of Mathematics. In Internet Encyclopedia of Philosophy, ISSN 2161-0002, <https://iep.utm.edu>
7. Maddy, P. (1990). *Realism in Mathematics*. Oxford: Oxford University Press.
8. Plato. (2013) *The Republic*. An Electronic Classics Series Publication.
9. Shapiro, S. (2005). Philosophy of Mathematics and its Logic: Introduction. In S. Shapiro (Ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic(pp.3-27)*. Oxford: University Press.

10. Shapiro, S. (2016). Mathematics in Philosophy, Philosophy in Mathematics: Three Case Studies. In F. Boccuni and A. Sereni (Eds.), *Objectivity, Realism, and Proof FilMat Studies in the Philosophy of Mathematics*. Springer International Publishing Switzerland.
11. Van Fraassen, B. (1980). *The Scientific Image*, Oxford: Clarendon Press.